CAMBRIDGE

Sixth Term Examination Papers **9475**
MATHEMATICS 3 Morning
Monday 24 June 2024 – Morning Time: 3 hours

Additional Material: Answer Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully.

Do **NOT** open this question paper until you are told that you may do so.

Read and follow the additional instructions on the front of the answer booklet.

INFORMATION FOR CANDIDATES

There are 12 questions in this paper.

Each question is marked out of 20.

You may answer as many questions as you choose. You are advised to concentrate on no more than six questions. Little credit will be given for fragmentary answers.

All your answers will be marked.

Crossed out work will NOT be marked.

Your final mark will be based on the six questions for which you gain the highest marks.

There is NO Mathematical Formulae Booklet.

Calculators are NOT permitted.

Bilingual dictionaries are NOT permitted.

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Section A: Pure Mathematics

1 Throughout this question, N is an integer with $N \geqslant 1$ and $S_N = \displaystyle\sum_{r=1}^{N} \frac{1}{r^2}$.

You may assume that $\displaystyle\lim_{N\to\infty} S_N$ exists and is equal to $\frac{1}{6}\pi^2$.

(i) Show that

$$\frac{1}{r+1} - \frac{1}{r} + \frac{1}{r^2} = \frac{1}{r^2(r+1)}.$$

Hence show that

$$\sum_{r=1}^{N} \frac{1}{r^2(r+1)} = \sum_{r=1}^{N} \frac{1}{r^2} - 1 + \frac{1}{N+1}.$$

Show further that $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)} = \frac{1}{6}\pi^2 - 1.$

(ii) Find $\displaystyle\sum_{r=1}^{N} \frac{1}{r^2(r+1)(r+2)}$ in terms of S_N, and hence evaluate $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)(r+2)}$.

(iii) Show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)^2} = \sum_{r=1}^{\infty} \frac{2}{r^2(r+1)} - 1.$$

2 **(i)** Solve the inequalities

(a) $\sqrt{4x^2 - 8x + 64} \leqslant |x + 8|\,$,

(b) $\sqrt{4x^2 - 8x + 64} \leqslant |3x - 8|\,$.

(ii) **(a)** Let $\text{f}(x) = \sqrt{4x^2 - 8x + 64} - 2(x - 1)$.

Show, by considering $\big(\sqrt{4x^2 - 8x + 64} + 2(x - 1)\big)\text{f}(x)$ or otherwise, that $\text{f}(x) \to 0$ as $x \to \infty$.

(b) Sketch $y = \sqrt{4x^2 - 8x + 64}$ and $y = 2(x - 1)$ on the same axes.

(iii) Find a value of m and the corresponding value of c such that the solution set of the inequality

$$\sqrt{4x^2 - 5x + 4} \leqslant |mx + c|$$

is $\{x : x \geqslant 3\}$.

(iv) Find values of p, q, m and c such that the solution set of the inequality

$$|x^2 + px + q| \leqslant mx + c$$

is $\{x : -5 \leqslant x \leqslant 1\} \cup \{x : 5 \leqslant x \leqslant 7\}$.

3 *Throughout this question, consider only $x > 0$.*

(i) Let

$$g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{x + c}{x(x + 1)}$$

where $c \geqslant 0$.

(a) Show that $y = g(x)$ has positive gradient for all $x > 0$ when $c \geqslant \frac{1}{2}$.

(b) Find the values of x for which $y = g(x)$ has negative gradient when $0 \leqslant c < \frac{1}{2}$.

(ii) It is given that, for all $c > 0$, $g(x) \to -\infty$ as $x \to 0$.

Sketch, for $x > 0$, the graphs of

$$y = g(x)$$

in the cases

(a) $c = \frac{3}{4}$,

(b) $c = \frac{1}{4}$.

(iii) The function f is defined as

$$f(x) = \left(1 + \frac{1}{x}\right)^{x+c}.$$

Show that, for $x > 0$,

(a) f is a decreasing function when $c \geqslant \frac{1}{2}$;

(b) f has a turning point when $0 < c < \frac{1}{2}$;

(c) f is an increasing function when $c = 0$.

4 **(i)** Show that if the acute angle between straight lines with gradients m_1 and m_2 is $45°$, then

$$\frac{m_1 - m_2}{1 + m_1 m_2} = \pm 1.$$

The curve C has equation $4ay = x^2$ (where $a \neq 0$).

(ii) If $p \neq q$, show that the tangents to the curve C at the points with x-coordinates p and q meet at a point with x-coordinate $\frac{1}{2}(p+q)$. Find the y-coordinate of this point in terms of p and q.

Show further that any two tangents to the curve C which are at $45°$ to each other meet on the curve $(y + 3a)^2 = 8a^2 + x^2$.

(iii) Show that the acute angle between any two tangents to the curve C which meet on the curve $(y + 7a)^2 = 48a^2 + 3x^2$ is constant. Find this acute angle.

5 In this question, $\mathbf{M}$ and $\mathbf{N}$ are non-singular 2×2 matrices.

The *trace* of the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as $\text{tr}(\mathbf{M}) = a + d$.

(i) Prove that, for any two matrices $\mathbf{M}$ and $\mathbf{N}$, $\text{tr}(\mathbf{MN}) = \text{tr}(\mathbf{NM})$ and derive an expression for $\text{tr}(\mathbf{M} + \mathbf{N})$ in terms of $\text{tr}(\mathbf{M})$ and $\text{tr}(\mathbf{N})$.

The entries in matrix $\mathbf{M}$ are functions of t and $\dfrac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}$ denotes the matrix whose entries are the derivatives of the corresponding entries in $\mathbf{M}$.

(ii) Show that

$$\frac{1}{\det \mathbf{M}} \frac{\mathrm{d}}{\mathrm{d}t} (\det \mathbf{M}) = \text{tr}\left(\mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} \right).$$

(iii) In this part, matrix $\mathbf{M}$ satisfies the differential equation

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \mathbf{MN} - \mathbf{NM},$$

where the entries in matrix $\mathbf{N}$ are also functions of t.

Show that $\det \mathbf{M}$, $\text{tr}(\mathbf{M})$ and $\text{tr}\left(\mathbf{M}^2\right)$ are independent of t.

In the case $\mathbf{N} = \begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$, and given that $\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ when $t = 0$, find $\mathbf{M}$ as a function of t.

(iv) In this part, matrix $\mathbf{M}$ satisfies the differential equation

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \mathbf{MN},$$

where the entries in matrix $\mathbf{N}$ are again functions of t.

The trace of $\mathbf{M}$ is non-zero and independent of t. Is it necessarily true that $\text{tr}(\mathbf{N}) = 0$?

6 **(i)** A particle moves in two-dimensional space. Its position is given by coordinates (x, y) which satisfy

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 3y + u$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y + u$$

where t is the time and u is a function of time. At time $t = 0$, the particle has position (x_0, y_0).

(a) By considering $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t}$, show that if the particle is at the origin $(0, 0)$ at some time $t > 0$, then it is necessary that $x_0 = y_0$.

(b) Given that $x_0 = y_0$, find a constant value of u that ensures that the particle is at the origin at a time $t = T$, where $T > 0$.

(ii) A particle whose position in three-dimensional space is given by co-ordinates (x, y, z) moves with time t such that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4y - 5z + u$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - 2z + u$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = x - 2y + u$$

where u is a function of time. At time $t = 0$, the particle has position (x_0, y_0, z_0).

(a) Show that, if the particle is at the origin $(0, 0, 0)$ at some time $t > 0$, it is necessary that y_0 is the mean of x_0 and z_0.

(b) Show further that, if the particle is at the origin $(0, 0, 0)$ at some time $t > 0$, it is necessary that $x_0 = y_0 = z_0$.

(c) Given that $x_0 = y_0 = z_0$, find a constant value of u that ensures that the particle is at the origin at a time $t = T$, where $T > 0$.

7 *In this question, you need not consider issues of convergence.*

For positive integer n let

$$\mathrm{f}(n) = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$$

and

$$\mathrm{g}(n) = \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} - \dots .$$

(i) Show, by considering a geometric series, that $0 < \mathrm{f}(n) < \dfrac{1}{n}$.

(ii) Show, by comparing consecutive terms, that $0 < \mathrm{g}(n) < \dfrac{1}{n+1}$.

(iii) Show, for positive integer n, that $(2n)!\,\mathrm{e} - \mathrm{f}(2n)$ and $\dfrac{(2n)!}{\mathrm{e}} + \mathrm{g}(2n)$ are both integers.

(iv) Show that if $q\mathrm{e} = \dfrac{p}{\mathrm{e}}$ for some positive integers p and q, then $q\,\mathrm{f}(2n) + p\,\mathrm{g}(2n)$ is an integer for all positive integers n.

(v) Hence show that the number e^2 is irrational.

8 **(i)** Explain why the equation $(y - x + 3)(y + x - 5) = 0$ represents a pair of straight lines with gradients 1 and -1. Show further that the equation

$$y^2 - x^2 + py + qx + r = 0$$

represents a pair of straight lines with gradients 1 and -1 if and only if $p^2 - q^2 = 4r$.

In the remainder of this question, C_1 is the curve with equation $x = y^2 + 2sy + s(s+1)$ and C_2 is the curve with equation $y = x^2$.

(ii) Explain why the coordinates of any point which lies on both of the curves C_1 and C_2 also satisfy the equation

$$y^2 + 2sy + s(s+1) - x + k(y - x^2) = 0$$

for any real number k.

Given that s is such that C_1 and C_2 intersect at four distinct points, show that choosing $k = 1$ gives an equation representing a pair of straight lines, with gradients 1 and -1, on which all four points of intersection lie.

(iii) Show that if C_1 and C_2 intersect at four distinct points, then $s < -\frac{3}{4}$.

(iv) Show that if $s < -\frac{3}{4}$, then C_1 and C_2 intersect at four distinct points.

Section B: Mechanics

9 The origin O of coordinates lies on a smooth horizontal table and the x- and y-axes lie in the plane of the table. A smooth sphere A of mass m and radius r is at rest on the table with its lowest point at the origin.

A second smooth sphere B has the same mass and radius and also lies on the table. Its lowest point has y-coordinate $2r\sin\alpha$, where α is an acute angle, and large positive x-coordinate.

Sphere B is now projected parallel to the x-axis, with speed u, so that it strikes sphere A. The coefficient of restitution in this collision is $\frac{1}{3}$.

(i) Show that, after the collision, sphere B moves with velocity

$$\begin{pmatrix} -\frac{1}{3}u\big(1+2\sin^2\alpha\big) \\ \frac{2}{3}u\sin\alpha\cos\alpha \end{pmatrix}.$$

(ii) Show further that the lowest point of sphere B crosses the y-axis at the point $(0, Y)$, where $Y = 2r\,(\cos\alpha\tan\beta + \sin\alpha)$ and

$$\tan\beta = \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}.$$

A third sphere C of radius r is at rest with its lowest point at $(0, h)$ on the table, where $h > 0$.

(iii) Show that, if $h > Y + 2r\sec\beta$, sphere B will not strike sphere C in its motion after the collision with sphere A.

(iv) Show that $Y < 2r\sec\beta$.

Hence show that sphere B will not strike sphere C for any value of α, if $h > \dfrac{8r}{\sqrt{3}}$.

10 A cube of uniform density ρ is placed on a horizontal plane and a second cube, also of uniform density ρ, is placed on top of it. The lower cube has side length 1 and the upper cube has side length a, with $a \leqslant 1$. The centre of mass of the upper cube is vertically above the centre of mass of the lower cube and all the edges of the upper cube are parallel to the corresponding edges of the lower cube. The contacts between the two cubes, and between the lower cube and the plane, are rough, with the same coefficient of friction $\mu < 1$ in each case. The midpoint of the base of the upper cube is X and the midpoint of the base of the lower cube is Y.

A horizontal force P is exerted, perpendicular to one of the vertical faces of the upper cube, at a point halfway between the two vertical edges of this face, and a distance h, with $h < a$, above the lower edge of this face.

(i) Show that, if the two cubes remain in equilibrium, the normal reaction of the plane on the lower cube acts at a point which is a distance

$$\frac{P(1+h)}{(1+a^3)\rho g}$$

from Y, and find a similar expression for the distance from X of the point at which the normal reaction of the lower cube on the upper cube acts.

The force P is now gradually increased from zero.

(ii) Show that, if neither cube topples, equilibrium will be broken by the slipping of the upper cube on the lower cube, and not by the slipping of the lower cube on the ground.

(iii) Show that, if $a = 1$, then equilibrium will be broken by the slipping of the upper cube on the lower cube if $\mu(1+h) < 1$ and by the toppling of the lower and upper cube together if $\mu(1+h) > 1$.

(iv) Show that, in a situation where $a < 1$ and $h\big(1 + a^3(1-a)\big) > a^4$, and no slipping occurs, equilibrium will be broken by the toppling of the upper cube.

(v) Show, by considering $a = \frac{1}{2}$ and choosing suitable values of h and μ, that the situation described in **(iv)** can in fact occur.

Section C: Probability and Statistics

11 *In this question, you may use without proof the results*

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n \text{ and } \sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}.$$

(i) Show that

$$r\binom{2n}{r} = (2n+1-r)\binom{2n}{2n+1-r}$$

for $1 \leqslant r \leqslant 2n$. Hence show that

$$\sum_{r=0}^{2n} r\binom{2n}{r} = 2\sum_{r=n+1}^{2n} r\binom{2n}{r}.$$

(ii) A fair coin is tossed $2n$ times. The value of the random variable X is whichever is the larger of the number of heads and the number of tails shown. If n heads and n tails are shown, then $X = n$.

Show that

$$\mathrm{E}(X) = n\left(1 + \frac{1}{2^{2n}}\binom{2n}{n}\right).$$

(iii) Show that $\dfrac{1}{2^{2n}}\dbinom{2n}{n}$ decreases as n increases.

(iv) In a game, you choose a value of n and pay £n; then a fair coin is tossed $2n$ times. You win an amount in pounds equal to the larger of the number of heads and the number of tails shown. If n heads and n tails are shown, then you win £n. How should you choose n to maximise your expected winnings per pound paid?

12 **(i)** A point is chosen at random in the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, so that the probability that a point lies in any region is equal to the area of that region. R is the random variable giving the distance of the point from the origin.

Show that the cumulative distribution function of R is given by

$$\mathrm{P}(R \leqslant r) = \sqrt{r^2 - 1} + \tfrac{1}{4}\pi r^2 - r^2 \cos^{-1}(r^{-1}),$$

when $1 \leqslant r \leqslant \sqrt{2}$. What is the cumulative distribution function when $0 \leqslant r \leqslant 1$?

(ii) Show that $\mathrm{E}(R) = \frac{2}{3}\displaystyle\int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2 - 1}}\,\mathrm{d}r$.

(iii) Show further that $\mathrm{E}(R) = \frac{1}{3}\left(\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right)$.

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